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Wigner Surmise for Domain Systems

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Abstract In random matrix theory the spacing distribution functions $p^{(n)}(s)$ are well fitted by the Wigner surmise and its generalizations. In this approximation the spacing functions are completely described by the behavior of the exact functions in the limits $s \to 0$ and $s \to \infty$. Most non equilibrium systems do not have analytical solutions for the spacing distribution and correlation functions. Because of that, we explore the possibility to use the Wigner surmise approximation in these systems. We found that this approximation provides a first approach to the statistical behavior of complex systems, in particular we use it to find an analytical approximation to the nearest neighbor distribution of the annihilation random walk.

Keywords Systems out of equilibrium · Random matrices · Wigner surmise

1 Introduction

The equilibrium and non equilibrium one dimensional systems are usually described by using the spacing distribution functions $p^{(n)}(s)$. This distribution give us the probability density function that the distance between two particles is s under the condition that between these particles there are n additional particles. In particular $p^{(0)}(s)$ is the nearest neighbor distribution, i.e., the probability density that the distance between two adjacent particles is s. In systems where we have domain formation $p^{(n)}(s)$ is the probability density function that the distance between two domain borders is s under the condition that between these borders there are n additional domain borders.

In Ref. [1] the authors study the statistical behavior of several out of equilibrium domain systems which evolve with formation of domains which grow in time. For intermediate times

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where the size of the domains is much smaller than the total size L of the system, the domain size distribution exhibit a dynamic scaling. The authors studied the statistical properties of these domains in the scaling regime. They found that the statistical behavior of those is similar to the one in random matrices, for example, the nearest neighbor distribution $p^{(0)}(s)$ of several out of equilibrium domain systems is well fitted by the Wigner surmise which also describe closely the eigenvalue spacing distribution $p^{(0)}(s)$ in the case of the circular (COE) and Gaussian (GOE) orthogonal ensembles in random matrix theory (actually this distribution is exact in the case of 2×2 matrices). However, the next distributions ($n > 0$) for these systems are different from their counterpart in random matrix theory. Another important aspect is the pair correlation function $g(r)$ which, in COE and GOE ensembles and the coalescing random walk and interacting random walk does not have any oscillation but in other systems $g(r)$ describe one oscillation near to $r = 1$. For more information see Refs. [1–4].

In random matrices theory we know the exact expressions for the spacing distribution functions, however, they are difficult to use, see Ref. [5]. However, in Ref. [6] the authors found an analytical approximation to $p^{(n)}(s)$. This approximation is called generalized Wigner surmise and provides an excellent fit to the exact spacing distribution functions.

In most non equilibrium domain systems, the main problem is the absence of analytical expressions for the spacing and correlation functions. In few cases we know the exact solution for $p^{(n)}(s)$, but, in the most of these few cases the solutions are too complicated to use.

Then, the question is: can we to extend the generalized Wigner surmise proposed in Ref. [6] to find a good approximation for $p^{(n)}(s)$ and $g(r)$ in the domain systems as it happens with the random matrix ensembles?

In order to answer this question in Sect. 2.1, we summarize the most important aspects of the generalized Wigner surmise, introduced in Ref. [6], applied to the Gaussian and Circular orthogonal ensembles of random matrices. In Sect. 2.2, we extend this method to apply it to the non equilibrium systems. This extension is straightforward, because we can apply this method to any system if we know the behavior of $p^{(n)}(s)$ for small and large values of s. In Sects. 3 to 6, we test this extension of the generalized Wigner surmise by applying it to several non equilibrium systems.

2 Generalized Wigner Surmise

2.1 Wigner Surmise for Random Matrices

In random matrix theory the analytic expressions for the spacing distribution functions of eigenvalues $p^{(n)}(s)$ in the circular and Gaussian orthogonal ensembles (COE and GOE respectively) in the limit of large matrices are given in terms of the eigenvalues μ_i and eigenfunctions $f_i(x)$ of the following integral equation, see Ref. [5]:

$$\mu_i f_i(x) = \int_{-1}^{1} e^{i\pi xys/2} f_i(y) dy. \tag{1}$$

The spacing distributions are calculated explicitly by using

$$E(2r, s) = \prod_{i=0}^{\infty}(1 - \lambda_{2i}) \sum_{0 \le j_1 < j_2 < \cdots < j_r} \prod_{i=1}^{r} \left(\frac{\lambda_{j_i}}{1 - \lambda_{j_i}} \right) \times \left[1 - (b_{j_1} + \cdots + b_{j_r}) \right], \tag{2}$$

$$E(2r-1,s)=\prod_{i=0}^{\infty}(1-\lambda_{2i})\sum_{0\leq j_1<j_2<\cdots<j_r}\prod_{i=1}^{r}\left(\frac{\lambda_{j_i}}{1-\lambda_{j_i}}\right)\times(b_{j_1}+\cdots+b_{j_r}),\tag{3}$$

where

$$b_j=f_{2j}(1)\int_{-1}^{1}f_{2j}(x)dx\Big/\int_{-1}^{1}f_{2j}^2(x)dx,\tag{4}$$

$$\lambda_j=s|\mu_j|^2/4,\tag{5}$$

and

$$p^{(n)}(s)=\frac{d^2}{ds^2}\sum_{j=0}^{n}(n-j+1)E(j,s).\tag{6}$$

These expressions are difficult to manage, however in Ref. [6], the authors find an excellent approximation for spacing distributions $p^{(n)}(s)$ from their well-known behavior in the limits $s\to 0$ and $s\to\infty$. This approximation is easy to use and provide an excellent fit to the exact distributions. We will use this approximation many times in this paper, because of that, we summarize now its most important aspects.

By definition, $p^{(n)}(s)$ is the probability density that an interval of length s which starts at a level contains exactly n levels and the next, the $n+1$ level, is in $[s,s+ds]$. In the same way, let $F^{(n)}(s)$ be the probability that an interval of length s which starts at a level, contains n levels. By using this definition we can write

$$F^{(n)}(s)=\int_{s}^{\infty}\left(p^{(n)}(s')-p^{(n-1)}(s')\right)ds'.\tag{7}$$

Additionally, let $r^{(n)}(s)$ be the probability density that an interval $[0,s]$ which starts at a level at $s=0$ is limited by a level on its right side, under the condition that there are exactly n levels in the interval $(0,s)$, i.e., $r^{(n)}(s)$ is the conditional probability

$$r^{(n)}(s)=\frac{p^{(n)}(s)}{F^{(n)}(s)},\tag{8}$$

this probability is called level repulsion function. Following Ref. [6], in the limit $s\to 0$, this equation can be written as

$$p^{(n)}(s)=r^{(n)}(s)\int_{0}^{s}p^{(n-1)}(s')ds'.\tag{9}$$

In the GOE ensemble the matrix elements are chosen using a Gaussian distribution, this fact suggest that $p^{(n)}(s)$ decays as Gaussian function. The appropriate function for fit is

$$p^{(n)}(s)=A_n s^{\alpha_n}e^{-B_n s^2},\tag{10}$$

under the surmise $r^{(n)}(s)\to s^{n+1}$ with $s\to 0$. Additionally, the functions $p^{(n)}(s)$ satisfy the normalization conditions

$$\int_{0}^{\infty}p^{(n)}(s)ds=1,\tag{11}$$

and

$$\int_0^\infty s p^{(n)}(s) ds = 1. \tag{12}$$

By using the surmise for the level repulsion and the normalization conditions, is straightforward to find [6]

$$A_n = 2\frac{B_n^{(\alpha_n+1)/2}}{\Gamma\left(\frac{\alpha_n+1}{2}\right)}, \tag{13}$$

$$B_n = \left[\frac{\Gamma\left(\frac{\alpha_n}{2}+1\right)}{(n+1)\Gamma\left(\frac{\alpha_n+1}{2}\right)}\right]^2, \tag{14}$$

where

$$\alpha_n = n + \frac{(n+1)(n+2)}{2}. \tag{15}$$

Then, the approximate spacing distribution functions $p^{(n)}(s)$ are given explicitly by

$$p^{(n)}(s) = \left[\frac{\Gamma\left(\frac{\alpha_n}{2}+1\right)}{(n+1)}\right]^{\alpha_n+1} \frac{2s^{\alpha_n}}{\Gamma\left(\frac{\alpha_n+1}{2}\right)^{\alpha_n+2}} e^{-\left[\frac{\Gamma\left(\frac{\alpha_n}{2}+1\right)}{(n+1)\Gamma\left(\frac{\alpha_n+1}{2}\right)}\right]^2 s^2}. \tag{16}$$

The result obtained for α_n coincides with the results obtained by using the exact expression for the spacing distribution functions, see Ref. [5]. Notice that the approximate spacing distributions functions are characterized by the level repulsion, normalization condition, scaling condition for the average spacing and Gaussian decay. This approximation is called generalized Wigner surmise and provides a very good approximation for $p^{(n)}(s)$, because it reproduce not only the distributions behavior in the limits $s \to 0$ and $s \to \infty$, but also reproduce their global behavior, as we can see in Fig. 1. In particular the function with $n = 0$ is called Wigner distribution. This fit allows us to calculate also the approximate pair correlation distribution $g(r)$. For this purpose we use

$$g(r) = \sum_{n=0}^{\infty} p^{(n)}(r), \tag{17}$$

then

$$g(r) = 2\sum_{n=0}^{\infty}\left[\frac{\Gamma\left(\frac{\alpha_n}{2}+1\right)}{(n+1)}\right]^{\alpha_n+1} \frac{r^{\alpha_n}}{\Gamma\left(\frac{\alpha_n+1}{2}\right)^{\alpha_n+2}} e^{-\left[\frac{\Gamma\left(\frac{\alpha_n}{2}+1\right)}{(n+1)\Gamma\left(\frac{\alpha_n+1}{2}\right)}\right]^2 r^2}. \tag{18}$$

In Fig. 1 we can see that this is a good approximation for $g(r)$, however, it is not as useful as the Wigner surmise for $p^{(0)}(s)$ because the exact expression for $g(r)$ is well-known and easy to use, see Ref. [5].

2.2 Wigner Surmise for Domain Systems

We can apply the method explained above to any system if we know the behavior of the spacing distribution functions that describe this system for small and large values of s. That means that we must know the level repulsion of $p^{(n)}(s)$ and its decay functional form.

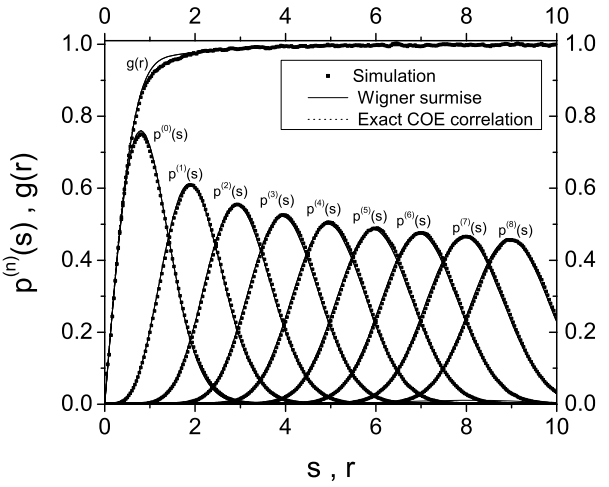

Fig. 1 Comparison between the generalized Wigner surmise and the COE ensemble. In the simulation we took 20000 matrices of size 200×200

For all systems considered in this paper $p^{(0)}(s)$ is well described by the Wigner distribution, because of that and following the method used in the random matrix theory we propose the next model

$$\alpha_n = \begin{cases} 1 & \text{for } n = 0 \\ h(n) & \text{for } n \geq 1, \end{cases} \tag{19}$$

with $h(n)$ is a function to determine which give us the level repulsion of the system. The spacing distribution functions in this model are given by

$$p^{(n)}(s) = \begin{cases} \frac{\pi}{2} s e^{-\frac{\pi}{4}x^2} & \text{if } n = 0 \\ A_n s^{\alpha_n} e^{-B_n x^{\beta_n}} & \text{if } n \geq 1, \end{cases} \tag{20}$$

using (11) and (12), we find

$$A_n = \frac{\beta_n B_n^{\frac{1+\alpha_n}{\beta_n}}}{\Gamma\left(\frac{1+\alpha_n}{\beta_n}\right)}, \tag{21}$$

and

$$B_n = \left(\frac{\Gamma\left(\frac{2+\alpha_n}{\beta_n}\right)}{(1+n)\Gamma\left(\frac{1+\alpha_n}{\beta_n}\right)} \right)^{\beta_n}, \tag{22}$$

where β_n characterize the long distance decay of the distribution function. For $\beta_n = 1$ we have Poisson decay and for $\beta_n = 2$ we have Gaussian decay. Note that our model is very similar to the one in Ref. [6], the only difference is that we write our equations in terms of a general level repulsion and we do not use necessarily Gaussian decay.

In this method we fit the exact spacing distribution functions to equation 20 by using its local behavior for small and large values of s. We shall call this the local fit.

We also can do the fit by using the entire behavior of the system, i.e., considering the whole interval $s \in [0, \infty[$. This method does not have physical meaning but we will use it in this paper in order to have a reference to compare. In next sections we will call it the global fit.

3 Independent Interval Approximation Model (IIA)

The independent interval approximation (IIA) is used as an approximate solution in many equilibrium and non equilibrium systems [1, 7, 8] in order to find analytical results. In particular, in Ref. [1] the IIA is used to find an approximate model for the statistical behavior of two non equilibrium systems which will be explained in next sections. In this approximation, $p^{(n)}(s)$ is given by the convolution product of $n+1$ nearest neighbor distribution factors, because of that, the spacing distribution functions can be calculated by using the Laplace transform, see Ref. [7]. However, in most cases we cannot compute explicitly the inverse Laplace transform in an analytical way for large values of n. Because of that, and taking into account that this method is commonly used to study many non equilibrium one dimensional systems, we propose to use the method exposed in previous section to find more tractable analytical approximate expressions to the distribution function $p^{(n)}(s)$ of the IIA method.

3.1 Independent Interval Model for Small Values of s

In Ref. [1] the authors choose $p^{(0)}(s)$ equal to the Wigner distribution. In order to apply the method of the last section, we need to know the behavior of $p^{(n)}(s)$ for small and large values of s. For the first region we expand the Wigner distribution in power series

$$p^{(0)}(s) = \frac{\pi}{2} s e^{-\frac{\pi}{4}s^2} = \frac{\pi}{2} s \left(1 - \frac{\pi}{4}s^2 + \cdots\right), \tag{23}$$

then, to the first order, the nearest neighbor distribution $p^{(0)}(s)$ has a lineal behavior, given by

$$p^{(0)}(s) \propto s. \tag{24}$$

In the same limit $s \to 0$, by using the independent interval approximation for arbitrary values of n, we have

$$p^{(n)}(s) \propto \int_{0<x_1<x_2\cdots<x_n<s} x_1(x_2 - x_1)\cdots(s - x_n)dx_1\cdots dx_n, \tag{25}$$

which can be evaluated by using the Laplace transform

$$\widetilde{p}^{(n)}(t) \propto \frac{1}{t^{2(n+1)}}, \tag{26}$$

and then, taking its inverse

$$p^{(n)}(s) \propto s^{2n+1}. \tag{27}$$

As consequence, in the IIA case the exponent α_n depends linearly on n

$$\alpha_n = 2n + 1. \tag{28}$$

By using this result it is possible to determine the behavior of the level repulsion function for $s \to 0$. Following Ref. [6] we have

$$r^{(n)}(s) \propto s^{f(n)}, \tag{29}$$

where $f(n)$ is the function to determine. By using (9), we can write

$$p^{(n)}(s) \propto s^{f(n)} \int_0^s p^{(n-1)}(s')ds', \tag{30}$$

then

$$p^{(n)}(s) \propto s^{f(n)+\cdots+f(0)+n}. \tag{31}$$

By comparing (27) with (31) is straightforward to find

$$f(n) = 1, \tag{32}$$

for all $n \geq 0$, as a consequence

$$r^{(n)}(s) \propto s, \quad s \to 0, \tag{33}$$

then, the level repulsion does not depend on n as it happens in the COE/GOE case.

3.2 Independent Interval Model for Large Values of s

Now, we need the behavior of $p^{(n)}(s)$ for large values of s. The exact expression for $p^{(n)}(s)$ is

$$p^{(n)}(s) = \int_{0<x_1<x_2\cdots<x_n<s} p^{(0)}(x_1)p^{(0)}(x_2-x_1)\cdots p^{(0)}(s-x_n)dx_1\cdots dx_n. \tag{34}$$

In our case $p^{(0)}(s)$ is given by the Wigner surmise,

$$p^{(0)}(s) = \frac{\pi}{2} s\, e^{-\pi s^2/4}, \tag{35}$$

then

$$p^{(n)}(s) = \left(\frac{\pi}{2}\right)^{n+1} \int_{0<x_1<x_2\cdots<x_n<s} x_1 e^{-\frac{\pi}{4}{x_1}^2}\cdots(s-x_n)e^{-\frac{\pi}{4}(s-x_n)^2}dx_1\cdots dx_n. \tag{36}$$

We can calculate the behavior of these functions for arbitrary values of n in the limit $s \to \infty$ as we show next. From Ref. [1] we know that at least the first two spacing distribution functions decay like Gaussian functions, then, we assume that for arbitrary values of n these functions have the form $p^{(n)}_{asy}(s) = M_n s^{\gamma_n} e^{-N_n s^2}$ in the limit $s \to \infty$. In order to eliminate the integrals in (36) we use the Laplace transform

$$\widetilde{p}^{(n)}(l) = \left(1 - l e^{l^2/\pi}\,\mathrm{erfc}\left(\frac{l}{\sqrt{\pi}}\right)\right)^{n+1}, \tag{37}$$

where $\mathrm{erfc}(z) = (2/\sqrt{\pi})\int_z^\infty e^{-t^2}\,dt$ is the complementary Gaussian error function. In the same way we take the Laplace transform in $p^{(n)}_{asy}$. Additionally, we expand both transforms in Taylor series around $l = 0$. Let be Z_j the jth coefficient in the expansion of (37) and Y_j is the one for the Laplace transform of $p^{(n)}_{asy}(s)$. We find that the coefficients of both expansions satisfy the relation $Y_i/Z_i = Y_j/Z_j$ in the limit $i, j \to \infty$. By using this method we can find M_n, N_n and γ_n. In fact we find that $N_n = \frac{\pi}{4n}$ and $\gamma_n = n+1$. In general, if we know $p^{(0)}(s)$ we can calculate the asymptotic behavior of $p^{(n)}(s)$ under the assumption that the IIA is valid for $s \to \infty$, but, as we will see in next sections, this is not always true.

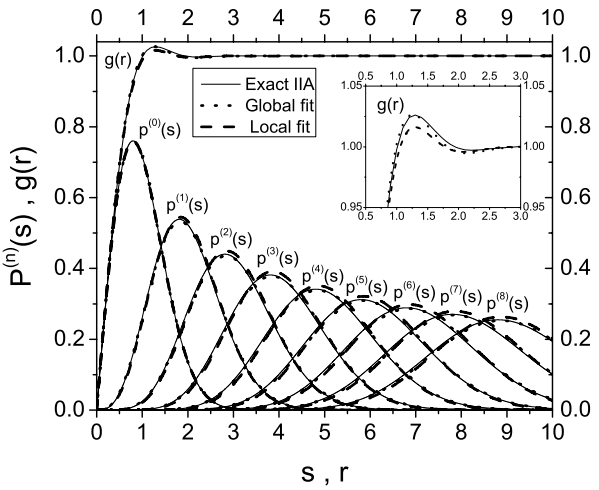

Fig. 2 Comparison between the exact statistical behavior of IIA, the generalized Wigner surmise (local fit) and the global fit

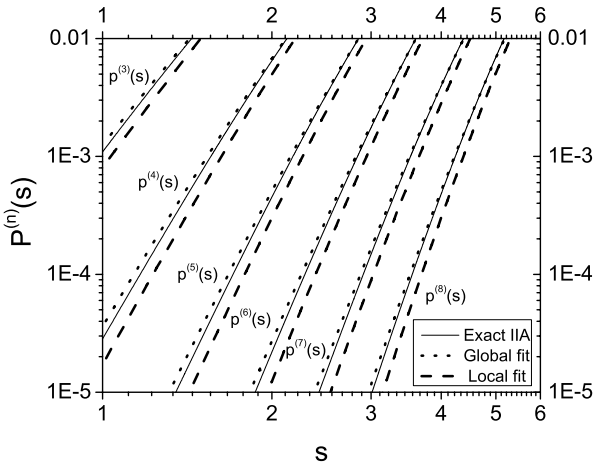

Fig. 3 Log-Log graphic for the spacing distribution functions for IIA

In the Fig. 2 we compare the exact statistical behavior of IIA with the generalized Wigner surmise, i.e., with a fit developed by using the behavior of $p^{(n)}(s)$ in the limits $s \to 0$ and $s \to \infty$, because of that from now on we will call it local fit. Also, we compare the global fit which was developed by using (19) to (22) and the complete behavior of $p^{(n)}(s)$ in the interval $[0, \infty[$. By using the values of α_n found in the global fit, we developed a new fit to determine the global behavior of α_n, explicitly in this case we have

$$\alpha_n = 1.8268n + 0.9954, \tag{38}$$

this result is close to the exact exponent (28), even when we use wrong functions in the fit; for example, the exact result for $p^{(1)}(s)$ is, see Ref. [1]

$$p^{(1)}(s) = \frac{\pi}{16} e^{-\frac{\pi s^2}{4}} \left(4s + \sqrt{2} e^{\frac{\pi s^2}{8}} \left(-4 + \pi s^2 \right) \operatorname{erf}\left(\frac{1}{2} \sqrt{\frac{\pi}{2}} s \right) \right), \tag{39}$$

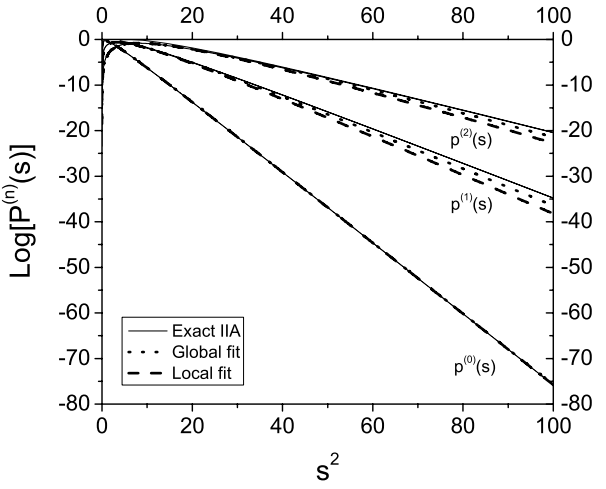

Fig. 4 Asymptotic behavior for $s \to \infty$ of the spacing distribution functions

which is very different form our surmise, however, both functions (20) and (39) have the same type of behavior in the limits $s \to 0$ and $s \to \infty$. Equation (18) for the correlation function it is still valid in both cases, global and local fit, we must only use (28) and (38) respectively. The main problem in the global fit approximation it is the use of non integer exponents in the level repulsion. Figure 3 show the differences between the three cases for small values of s, naturally in this region the graph of the global fit is not parallel to graph of the exact result as it actually happens in the local fit approximation. In Fig. 4 we can see the linear behavior of $p^{(n)}(s)$ in limit $s \to \infty$, which implies that the distribution functions decay like a Gaussian function as it was to be expected.

4 Coalescing and Annihilation Random Walks

4.1 Coalescing Random Walk (CRW)

In the coalescing random walk, the particles describe independent random walks along a one dimensional lattice and they are subjected to the reaction $A + A \to A$. This system is well studied [2, 9–11] and its analytical solution is well-known, because of that, it is used as an approximation to more complex systems. Let $q^{(n)}(s)$ be the conditional probability that given one particle its next neighbor is at a distance s. From its definition $q^{(n)}(s)$ is given by

$$q^{(n)}(s) = \int_{0<y_1\cdots<y_n<s} \omega^{(n+2)}(0, y_1, \ldots, y_n, s) dy_1 \cdots dy_n, \tag{40}$$

with

$$\omega^{(n)}(x_1, \ldots, x_n) = -\frac{\partial^n E^{(n-1)}(x_1, y_1, \ldots, x_{n-1}, y_{n-1})}{\partial x_1 \cdots \partial x_{n-1} \partial y_{n-1}}\bigg|_{y_1=x_2,\ldots,y_{n-1}=x_n}, \tag{41}$$

$$E^{(n)}(x_1, y_1, \ldots, x_n, y_n, t) = \sum_{p=1}^{(2n-1)!!} \sigma_p E^{(1)}(z_{1,p}, z_{2,p}, t) \cdots E^{(1)}(z_{2n-1,p}, z_{2n,p}, t), \tag{42}$$

where $z_{1,p}, z_{2,p}, \ldots, z_{2n,p}$ symbolize an ordered permutation, p, of the variables $x_1, y_1, \ldots, x_n, y_n$, such that

$$z_{1,p} < z_{2,p}, z_{3,p} < z_{4,p}, \ldots, z_{2n-1,p} < z_{2n,p}, \tag{43}$$

and

$$z_{1,p} < z_{3,p} < z_{5,p} \cdots < z_{2n-1,p}. \tag{44}$$

The function $E^{(1)}(x_1, y_1, t)$ is the probability that from x_1 to y_1 the lattice is empty at time t. Then it is possible generate the complete solution for the CRW from $E^{(1)}(x_1, y_1, t)$, which is given by the solution of the diffusion equation under the suitable boundary conditions (see Ref. [2]). In fact, the exact expression for this function is

$$E^{(1)}(x_1, y_1, t) = \operatorname{erfc}\left(\frac{y_1 - x_1}{\sqrt{8Dt}}\right), \tag{45}$$

with D the diffusion constant and t the time, for additional information see Ref. [2]. For practical purposes, the solution given by (40) to (45) is hard to evaluate for arbitrary values of n but it can be evaluated in the limit $s \to 0$ using Taylor series. The case $n = 0$ is trivial, the Taylor expansion for (45) is

$$E^{(1)}(x_1, y_1, t) = 1 - \frac{y_1 - x_1}{\sqrt{2\pi}(Dt)^{1/2}} + \frac{(y_1 - x_1)^3}{24\sqrt{2\pi}(Dt)^{3/2}} - \frac{(y_1 - x_1)^5}{640\sqrt{2\pi}(Dt)^{5/2}} + O(x, y)^7, \tag{46}$$

then

$$q^{(0)}(x_2, x_1) = \omega^{(2)}(x_1, x_2) = -\frac{\partial^2}{\partial x_1 \partial y_1} E^{(1)}(x_1, y_1, t)\Big|_{y_1 = x_2}, \tag{47}$$

$$q^{(0)}(x_2, x_1) = \frac{x_2 - x_1}{4\sqrt{2\pi}(Dt)^{3/2}} - \frac{(x_2 - x_1)^3}{32\sqrt{2\pi}(Dt)^{5/2}} + O(x)^5. \tag{48}$$

Making the variable change $s = \frac{x_2 - x_1}{\sqrt{2\pi Dt}}$ and taking into account that $p^{(0)}(s) = 2\pi Dt\, q^{(0)}(x_2, x_1)$, the product Dt disappears (dynamical scaling) in the above equation. Then, to first order, we have

$$p^{(0)}(s) = \frac{s\pi}{2} + O(s)^3. \tag{49}$$

For small values of s, $p^{(0)}(s)$ has a linear behavior, i.e., $\alpha_0 = 1$. The case $n = 1$ is more complicated, in fact we have

$$\omega^{(3)}(x_1, x_2, x_3) = -\frac{\partial^3}{\partial x_1 \partial x_2 \partial y_2} E^{(2)}(x_1, y_1, x_2, y_2, t)\Big|_{y_1 = x_2, y_2 = x_3}, \tag{50}$$

where

$$\begin{aligned} E^{(2)}(x_1, y_1, x_2, y_2, t) = {} & E(x_1, y_1, t)E(x_2, y_2, t) + E(x_1, y_2, t)E(y_1, x_2, t) \\ & - E(x_1, x_2, t)E(y_1, y_2, t), \end{aligned} \tag{51}$$

then

$$\omega^{(3)}(x_1, x_2, x_3) = \frac{(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)}{32\pi(Dt)^3} + O(x)^4, \tag{52}$$

in that way $q^{(1)}(x_1, x_3, t)$ is given by

$$q^{(1)}(x_3, x_1, t) = \int_{x_1}^{x_3} \frac{(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)}{32\pi(Dt)^3} dx_2 + O(x)^5. \tag{53}$$

Integrating

$$q^{(1)}(x_3, x_1, t) = \frac{(x_3 - x_1)^4}{192\pi(Dt)^3} + O(x)^5. \tag{54}$$

Using again the variable change, it is straightfoward to find

$$p^{(1)}(s) = \frac{\pi^2 s^4}{24} + O(x)^5, \tag{55}$$

we conclude that $\alpha_1 = 4$. In general for an arbitrary value of n, we find that the first term in the expansion is

$$q^{(n)}(x_1, x_n, t) \propto \int_{x_1}^{x_n} \cdots \int_{x_1}^{x_3} \prod_{1 \le i < j \le n} (x_j - x_i) dx_2 \cdots dx_{n-1}, \tag{56}$$

therefore, the above equation has $(n+1)(n+2)/2$ different factors which implies that the integrand is proportional to $x_i^{(n+1)(n+2)/2}$. Making the integral and the usual variable change, the final expression for small values of s is proportional to $s^{(n+1)(n+2)/2+n}$, explicitly, we have

$$\alpha_n = n + \frac{(n+1)(n+2)}{2}. \tag{57}$$

This is the same result reported in Ref. [6] for the GOE/COE case and coincides with the partial result presented in Ref. [11] for the CRW. We made again both fits, global and local. The global fit was made with the data from our simulation where we use a lattice with 1000 sites and 500 particles in $t = 0$. The data to build the histograms was taken at three different times $T = 50$, $T = 100$ and $T = 200$ over 50000 realizations. In this case the global fit is not as accurate as in the IIA case as we can see in Figs. 5, 6 and 7, but it still is a good approximation. We use again (19) to (22); and additionally we supposed a Gaussian decay ($\beta_n = 2$). The global fit gives

$$\alpha_n = 2.8688n + 0.8621. \tag{58}$$

The global fit gives an erroneous exponent which depend linearly with n, this result it does not coincide with the analytical result (57), where, α_n is a quadratic function of n. The local fit it is very different from the simulation results and coincides with the statistical behavior of the COE/GOE ensembles.

4.2 Annihilation Random Walk

In the annihilation random walk, the particles describe independent random walks along the one dimensional lattice but they are now subjected to the annihilation reaction $A + A \to 0$ when two particles meet at the same site. It is known that the nearest neighbor distribution $p_{ann}^{(0)}(s)$ has a very different behavior from the Wigner distribution considered in the other models presented here. In particular, $p_{ann}^{(0)}(s)$ has an exponential decay for large s instead of a Gaussian decay.

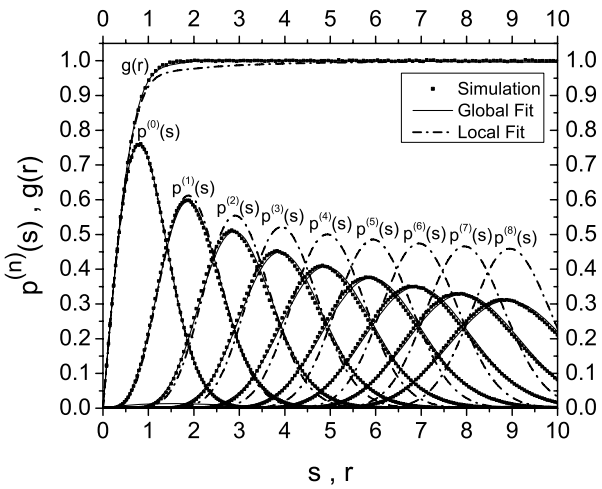

Fig. 5 Comparison between the statistical behavior of CRW, the global fit and the local fit

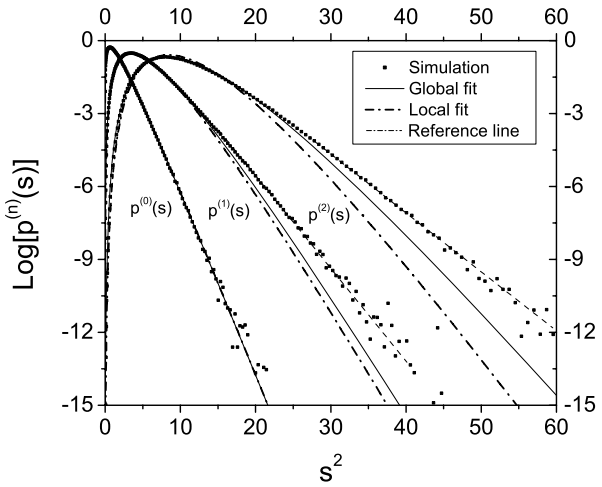

Fig. 6 Log-Log graphic for the spacing distribution functions of the CRW

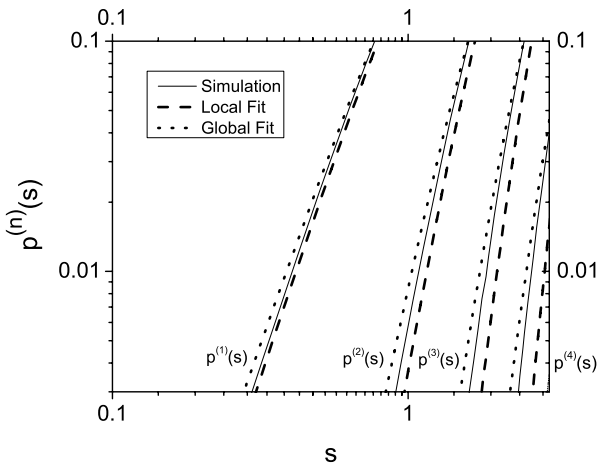

Fig. 7 Statistical behavior of CRW for small values of *s*

However, we can use the local or the global fit for $p^{(n)}(s)$ of the coalescing random walk found in the previous section to obtain $p^{(0)}_{ann}(s)$ of the annihilation random walk. This can be done using the relation found in Ref. [11]

$$p^{(0)}_{ann}(s) = \sum_{n\geq 0} \frac{1}{2^n} p^{(n)}(2s). \tag{59}$$

For short distances $s \to 0$, using the local fit for the distribution functions $p^{(n)}(s)$ of the CRW, found in the previous section, into (59), we obtain

$$p^{(0)}_{ann}(s) = \pi s - \pi^2 s^3 + \frac{65536}{729\pi^3} s^4 + \frac{\pi}{2} s^5 + O(s^6), \quad s \to 0, \tag{60}$$

which favorably compares to the exact result derived by Derrida and Zeitak [12]

$$p^{(0)}_{ann,\text{exact}}(s) = \pi s - \pi^2 s^3 + \frac{\pi^2}{3} s^4 + \frac{\pi^3}{2} s^5 + O(s^6), \quad s \to 0. \tag{61}$$

Up to order 5, only the coefficient of order 4 differs, but nevertheless with a numerical difference of only 12%,

$$\frac{\pi^2}{3} s^4 \approx 3.30 s^4 \quad \text{and} \quad \frac{65536}{729\pi^3} s^4 \approx 2.90 s^4. \tag{62}$$

In comparison, the result obtained in Ref. [8] using the IIA, gives a much worse approximation than ours,

$$p^{(0)}_{ann,\text{IIA}}(s) = \pi s - \frac{5}{6}\pi^2 s^3 + 0 + \frac{49\pi^3}{120} s^5 + O(s^6), \quad s \to 0. \tag{63}$$

In order to test numerically the validity of our approximations for any value of s, we implement a simulation for the annihilation random walk for a one dimensional lattice with 2000 sites, 100 particles at $t = 0$ over 20000 realizations, the histogram was build using three times $T = 1000$, $T = 1500$ and $T = 2000$. By using the global and the local fit for the distribution functions $p^{(n)}(s)$ of the CRW, with (59), we obtain two numeric approximations for $p^{(0)}_{ann}(s)$ of the annihilation random walk (numerically, we summed 200 terms in expression (59)).

We can see in Fig. 8 that the global and local fits provide a good approximation for the nearest neighbor distribution of the annihilation random walk. Additionally, Fig. 9 compare the long distance behavior of $p^{(0)}_{ann}(s)$ obtained by our method with the global and the local fit, with the asymptotic exact result

$$p^{(0)}_{ann,\text{exact}}(s) \approx 1.8167 e^{-1.3062 s}, \quad s \to \infty, \tag{64}$$

given in Ref. [12]. In the same figure, it is also compared to the IIA approximation of Ref. [8], rederived in Ref. [11],

$$p^{(0)}_{ann,\text{IIA}}(s) \approx 1.6777 e^{-1.2685 s}, \quad s \to \infty, \tag{65}$$

and an "improved IIA" approximation proposed in Ref. [11], based on (59) and taking into account correlations of third order in the distribution functions $p^{(0)}(s)$ of the CRW,

$$p^{(0)}_{ann,\text{iIIA}}(s) \approx 1.7290 e^{-1.2853 s}, \quad s \to \infty. \tag{66}$$

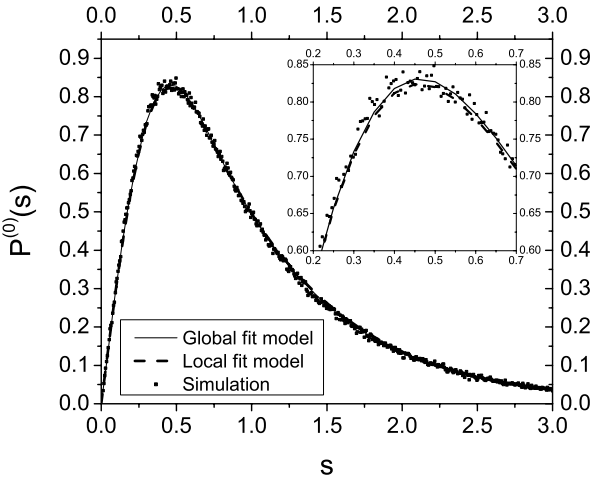

Fig. 8 Approximation for $p_{ann}^{(0)}(s)$ by using global and local fits

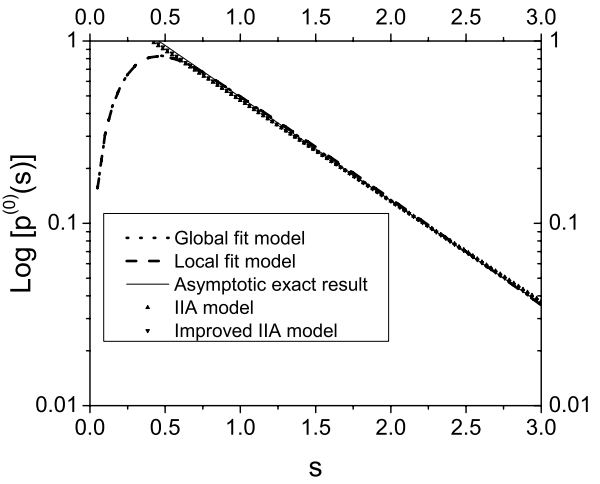

Fig. 9 Asymptotic behavior of $p_{ann}^{(0)}(s)$

From Fig. 9 it is clear that our approximations for $p_{ann}^{(0)}(s)$ exhibit an exponential decay for large s: $p_{ann}^{(0)}(s) \approx a\exp(-bs)$. Numerically, we evaluated, for the local fit,

$$a = 1.8892 \pm 0.0082, \quad b = 1.3193 \pm 0.0021, \tag{67}$$

and, for the global fit,

$$a = 1.84733 \pm 0.0014, \quad b = 1.31367 \pm 0.00036. \tag{68}$$

Our approximation give results more close to the exact result (64) than the ones of (65) and (66).

5 Spin System

This system was introduced in Ref. [3], where the authors consider a chain of L Ising spins with nearest neighbor ferromagnetic interaction J. The chain is subject to spin-exchange dynamics with a driving force E that favors motion of up spins to the right over motion to the left. After an initial transient regime, domains of spins up and domains of spin down form. These domains grow in time exhibiting a scaling behavior. We study the scaled domain size distribution $p^{(0)}(s)$, and more generally the spacing distribution functions of the edges of the domains $p^{(n)}(s)$ as defined in the introduction.

For this system, we do not have an analytical expression for the spacing distribution functions, because of that, we must start exploring numerically the behavior of $p^{(n)}(s)$ for small and large values of s. In Fig. 10, we can see the linear behavior of the spacing distribution function for $s \to 0$. Using values in this region we develop a fit which suggest that $\alpha_1 = 3$ and $\alpha_2 = 6$ approximately. Naturally $\alpha_0 = 1$, however it is very difficult to know using this method the next exponents because it is not possible develop a numerical simulation with enough precision.

Curiously, these exponents for $n = 0, 1, 2$ are given by the equation

$$\alpha_n = \frac{(n+1)(n+2)}{2}, \tag{69}$$

which is very similar to its counterpart in COE and GOE cases. For $s \to \infty$, $p^{(n)}(s)$ decays like a Gaussian function as we can see in Fig. 11. In this case the global fit gives

$$\alpha_n = 1.270n + 0.920. \tag{70}$$

In Fig. 12, we show the results given by (19) to (22) for the global fit in comparison with the simulation results which was made with a lattice with 1000 sites, equal number of spins up and down taken at two times $t = 34$ and $t = 48$ to build the histograms. The result for $g(r)$ is very good with a maximum error of 2.5%. Unfortunately this approximation is not good enough for $p^{(n)}(s)$ but at least it reproduce qualitatively the behavior of the real functions for $s \to \infty$. The local fit gives terrible results as it happens in the CRW case.

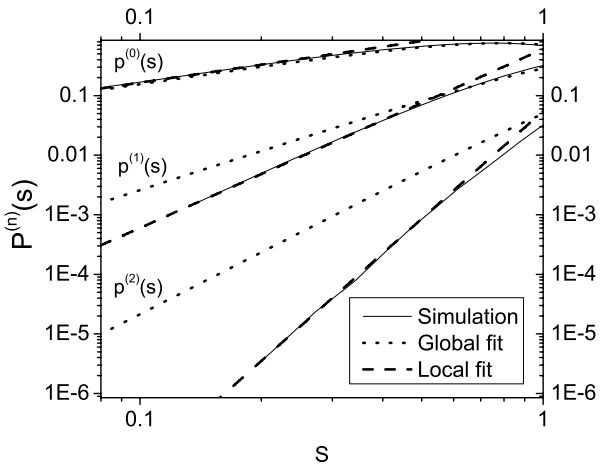

Fig. 10 Log-Log graphic for the spacing distribution functions of spin system

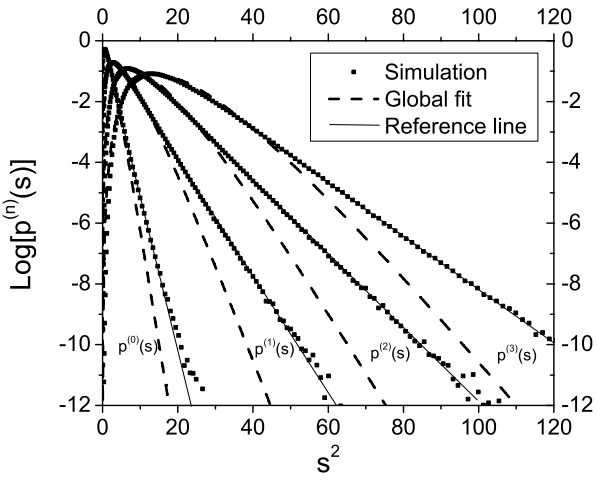

Fig. 11 Asymptotic behavior of the spin system

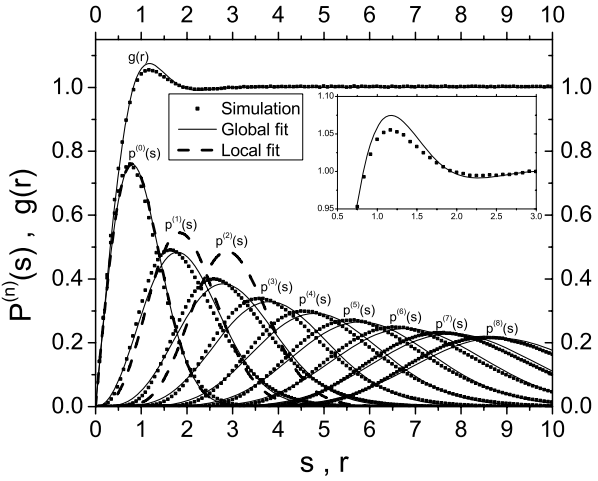

Fig. 12 Comparison between the statistical behavior of the spin system and the global fit

6 Gas System

This system was originally studied in Ref. [4]. There, the authors studied the biased diffusion of two species in a fully periodic $2 \times L$ rectangular lattice half filled with two equal number of two types of particles (labeled by their charge $+$ or $-$). An infinite external field drives the two species in opposite directions along the x axis (long axis). The only interaction between particles is an excluded volume constraint, i.e., each lattice site can be occupied at most by only one particle. This system also has a scaling regime with formation of domains filled with particles (regardless of their type) and of empty domains. It is convenient to consider an effective one-dimensional coarse grained version of the system, using the method explained originally in Ref. [4]. We are interested in the scaled spacing distribution functions of the edges of the domains of the effective one-dimensional coarse grained version of the system.

As it happens in the spin system, we do not know the analytic expression of the spacing and pair correlation functions. We follow the same numeric method used in the spin system. In Fig. 13, we can see the linear behavior of $p^{(n)}(s)$ which, by fit, give us $\alpha_1 = 3$ and $\alpha_2 = 5$

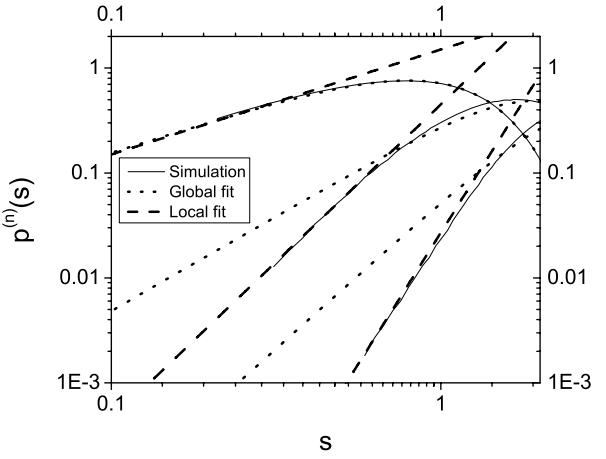

Fig. 13 Log-Log graphic for the spacing distribution functions of gas system

approximately, and of course $\alpha_0 = 1$. This fact suggest a linear behavior for α_n given by

$$\alpha_n = 2n + 1 \tag{71}$$

but again we could not find the next exponents with enough precision in order to validate the above equation. For $s \to \infty$ we found that $p^{(0)}(s)$ decays like a Gaussian function ($\beta_0 = 2$), but for $n > 0$, we found that β_n is an indeterminate function of n. For example, in Fig. 14, we can see the asymptotic behavior for two consecutive spacing distribution functions, the figure suggest $\beta_0 = 2$ for $p^{(0)}(s)$ and $\beta_1 \neq 2$ for $p^{(1)}(s)$ as it happens in Ref. [13]. Because it is difficult determine the exact value of β_n from the graphics, we implement a linear regression to find which value of β_n give us a better "straight" line. With this method we find, for example, that $\beta_1 = 2.6$ for $n = 1$, $\beta_5 = 3$ for $n = 5$ and $\beta_8 = 3.2$ for $n = 8$. In the linear regressions we took values between $5.5 \geq s \geq 2.5$, $11.7 \geq s \geq 7$ and $15.5 \geq s \geq 10$ respectively. Because of that, for the gas system we propose a model where β_n depends on n. In particular, we choose $\beta_n = 2.6 + 0.1(n - 1)$. With this model, the global fit gives

$$\alpha_n = 1.016n + 0.788. \tag{72}$$

The results of the global fit are shown in Fig. 15, again we find good fit for $g(r)$ with a maximum error of 2% approximately but the agreement for $p^{(n)}(s)$ is not so good. Additionally, we include the first spacing distribution obtained with the local fit and our model for β_n.

7 Conclusion

In COE and GOE ensembles, the spacing distribution functions $p^{(n)}(s)$ can be well described by using their behavior for small and large values of s (local fit) as it happens in IIA case, however, this is not true for more complex systems like CRW, spin and gas systems. This result was to be expected because in general the spacing distribution functions are characterized also by their inter medium behavior. In general, the global fit gives better results in comparison with the local fit but it fails to reproduce the level repulsion, in fact, gives non integer exponents. The level repulsion for the CRW has the same behavior that the circular and Gaussian orthogonal ensembles, i.e., both systems are equivalents for $s \to 0$.

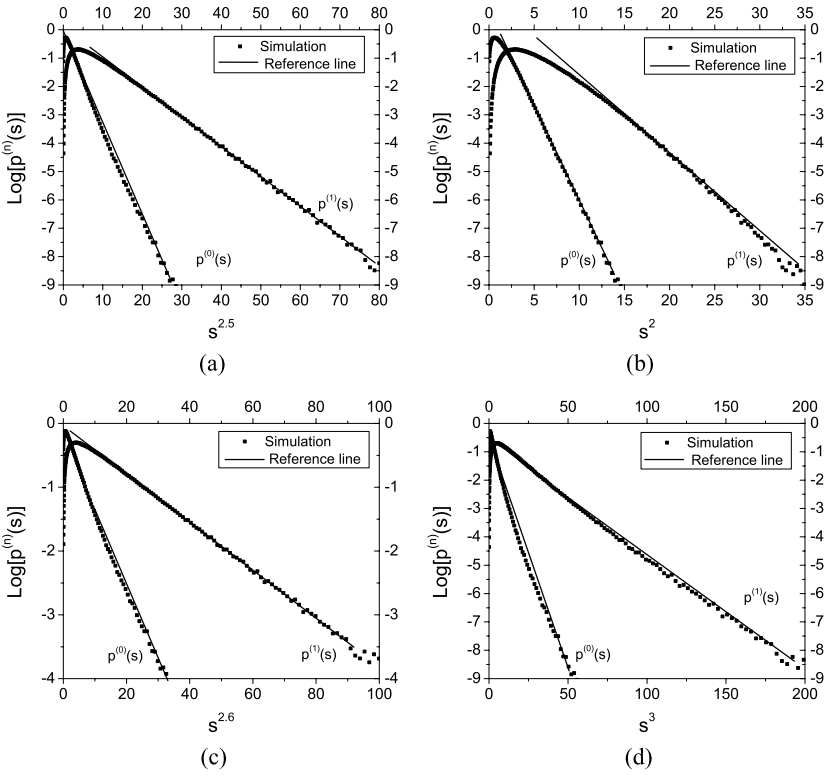

Fig. 14 Asymptotic behavior of $p^{(0)}(s)$ and $p^{(1)}(s)$ for the gas system

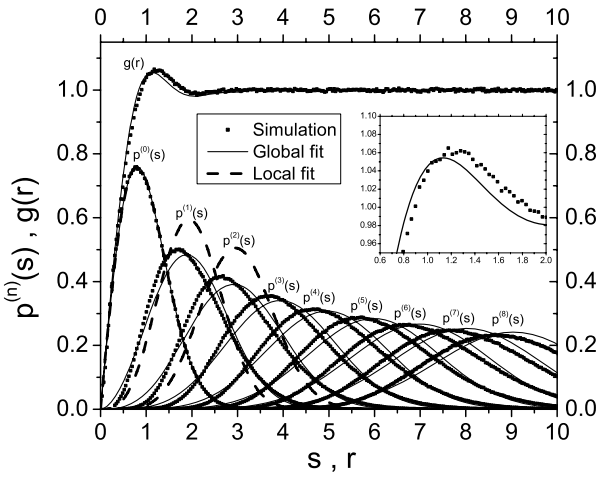

Fig. 15 Comparison between the statistical behavior of the gas system and the global fit

The numerical results suggest that the IIA and the gas system are also equivalents in that region. We find numerical evidence that the spacing distributions functions for gas system is described by a non universal function, in fact, they decay as $M_n s^{\gamma_n} e^{-N_n\ s^{\beta_n}}$ for $n > 0$, with

β_n an indeterminate function of n. In general the global and local fit provides a first approximation for $p^{(n)}(s)$ and $g(r)$, which can be used as a good approximation as it happens in the annihilation random walk case. These approximations also serve to classify the spacing distribution functions according to their level of repulsion and their decay functional form.

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